THE THRESHOLD BETWEEN EFFECTIVE AND NONEFFECTIVE DAMPING FOR SEMILINEAR WAVE EQUATIONS

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Abstract. In this paper we study the global existence of small data solutions to the Cauchy problem

$$u_{tt}-\triangle u+\frac{\mu}{1+t}\,u_t=f(t,u)\,,\qquad u(0,x)=u_0(x)\,,\quad u_t(0,x)=u_1(x)\,,$$

where $\mu\geq 2$. We obtain estimates for the solution and its energy with the same decay rate of the linear problem. We extend our results to a model with polynomial speed of propagation, and to a model with an exponential speed of propagation and a constant damping $\nu\,u_t$.

1. Introduction

The classical semilinear damped wave equation

$$\begin{cases} u_{tt}-\triangle u+u_t=f(u), & t\geq 0\,,\ x\in\mathbb{R}^n\,,\\ u(0,x)=u_0(x)\,, \\ u_t(0,x)=u_1(x)\,, \end{cases} \tag{1}$$

has been deeply investigated. In particular, if we assume small, compactly supported data, then by using some linear decay estimates [17] one can prove that there exists a global solution to (1) if $p>1+2/n$, and $p\leq 1+2/(n-2)$ if $n\geq 3$ (see [22]). This exponent is *critical*, that is, for suitable nontrivial, arbitrarily small data and $f(u)=|u|^p$ with $1<p\leq 1+2/n$, there exists no global solution to (1) (see [22, 31]).

If one removes the compactness assumption on the data, still one may obtain global existence for $p>1+2/n$ if the data are small in the norm of the energy space ($H^1\times L^2$) and in the L^1 norm in space dimension $n=1,2$ (see [9]). In space dimension $n\geq 3$ the compactness assumption on the data may be replaced by assuming that the data are small in the energy space with a suitable weight [11].

On the other hand, weakening the assumption of smallness replacing the L^1 norm of the data with the L^m norm for some $m\in(1,2)$, the *critical* exponent becomes $1+2m/n$ (see [10]). In particular, one obtains $1+4/n$ if the smallness is only taken in the energy space, without additional L^m regularity or compact support assumption. The same exponent was first obtained in [20] by using a *modified potential well* technique.

It has been recently proved [4] that the exponent $1+2/n$ remains *critical* if we consider the wave equation with a time-dependent *effective* damping $b(t)u_t$ satisfying suitable assumptions. We say that the damping term is *effective* for the wave equation if the linear estimates have the same decay rate of the corresponding heat equation $b(t)u_t-\triangle u=0$ (see [26, 28, 29, 30]). In fact, the exponent $1+2/n$ was first proved to be critical by Fujita for the semilinear heat equation [7].

In the special case $b(t)=\mu\,(1+t)^{-k}$, the dissipation is *effective* for any $\mu>0$, if $|\kappa|<1$. In this

2010 *Mathematics Subject Classification.* 35L71 Semilinear second-order hyperbolic equations.
Key words and phrases. semilinear equations, global existence, damped waves, scale-invariant, propagation speed, critical exponent.

special case, a global existence result has been obtained in [16, 19]. On the other hand, if $b(t)$ is a sufficiently smooth function satisfying $\limsup_{t\to\infty} tb(t) < 1$ then the dissipation is *non effective* [27]. The case $b(t) = \mu(1+t)^{-1}$ with $\mu \geq 1$ is more difficult to manage, since the dissipation is *effective* for large μ and *noneffective* for small μ. The precise threshold depends on which type of estimate one is studying.
Completely different effects appear if one consider a space-dependent damping term [12, 13, 18] or a time-space dependent damping term [15, 23]; in this case the exponent for the global existence changes accordingly to the decay in the space variable.

In this paper, we consider the Cauchy problem

$$\begin{cases} u_{tt} - \Delta u + \frac{\mu}{1+t}\, u_t = f(t,u), \quad t \geq 0,\ x \in \mathbb{R}^n\,, \\ u(0,x) = u_0(x)\,, \\ u_t(0,x) = u_1(x)\,. \end{cases} \tag{2}$$

Hypothesis 1. We assume that

$$f(t,0) = 0\,, \qquad \text{and} \quad |f(t,u) - f(t,v)| \lesssim (1+t)^\gamma\, |u-v|(|u|+|v|)^{p-1}\,, \tag{3}$$

for some $\gamma \geq -2$ and $p > 1$, satisfying $p \leq 1 + 2/(n-2)$ if $n \geq 3$.

Notation 1. We will use the following notation.

- We say that there exists *a solution to* (2), if there exists a unique
$$u \in \mathcal{C}([0,\infty), H^1) \cap \mathcal{C}^1([0,\infty), L^2)\,,$$
global solution to (20), in a weak sense.
- We refer to
$$\|(\nabla u, u_t)(t,\cdot)\|_{L^2}^2 := \|\nabla u(t,\cdot)\|_{L^2}^2 + \|u_t(t,\cdot)\|_{L^2}^2\,,$$
as the energy of the solution to (2).
- For any $m \in [1,2)$ we define
$$\mathcal{D}_m := (L^m \cap H^1) \times (L^m \cap L^2)\,, \qquad \|(u,v)\|_{\mathcal{D}_m}^2 := \|u\|_{L^m}^2 + \|u\|_{H^1}^2 + \|v\|_{L^m}^2 + \|v\|_{H^1}^2\,.$$

For the ease of reading, we collect our main results them in three separate theorems.

Theorem 1. *Let $n \geq 1$, $\mu \geq 2$ and $p > 1 + 2(2+\gamma)/n$. Then there exists $\epsilon > 0$ such that for any initial data*

$$(u_0,u_1) \in H^1 \times L^2\,, \qquad \textit{satisfying} \quad \|(u_0,u_1)\|_{H^1\times L^2} \leq \epsilon\,, \tag{4}$$

there exists a solution to (2)*. Moreover, the solution and its energy satisfy the estimates*

$$\|u(t,\cdot)\|_{L^2} \lesssim \|(u_0,u_1)\|_{H^1\times L^2}\,, \tag{5}$$

$$\|(\nabla u, u_t)(t,\cdot)\|_{L^2} \lesssim (1+t)^{-1}\, \|(u_0,u_1)\|_{H^1\times L^2}\,. \tag{6}$$

Theorem 2. *Let $n \leq 4$, $\mu \geq n+2$ and*

$$p > 1 + (2+\gamma)/n\,,$$

if $\gamma \geq n-2$, or $p \geq 2$ otherwise. Then there exists $\epsilon > 0$ such that for any initial data

$$(u_0,u_1) \in \mathcal{D}_1\,, \qquad \textit{satisfying} \quad \|(u_0,u_1)\|_{\mathcal{D}_1} \leq \epsilon\,, \tag{7}$$

there exists a solution to (2). *Moreover, the solution and its energy satisfy the decay estimates*

$$\|u(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n}{2}}\,\|(u_0,u_1)\|_{\mathcal{D}_1}\,, \tag{8}$$

$$\|(\nabla u,u_t)(t,\cdot)\|_{L^2} \lesssim \begin{cases} (1+t)^{-\frac{n}{2}-1}\,\|(u_0,u_1)\|_{\mathcal{D}_1} & \text{if } \mu>n+2,\\ (1+t)^{-\frac{\mu}{2}}\log(e+t)\,\|(u_0,u_1)\|_{\mathcal{D}_1} & \text{if } \mu=n+2. \end{cases} \tag{9}$$

The exponent $1+(2+\gamma)/n$ in Theorem 2 can be proved to be *critical* by using a *modified* test function method, that is, there exists no global solution to (2) if $p\le 1+(2+\gamma)/n$, for suitable data, arbitrarily small in $\mathcal{D}_1$ (see Example 2 in [3]).

Theorem 2 is a special case of the following.

Theorem 3. *Let $m\in[1,2)$, $n\le 4/(2-m)$,*

$$\mu \ge 2+n\left(\frac{2}{m}-1\right), \quad \textit{and} \tag{10}$$

$$p>1+\frac{m(2+\gamma)}{n}, \tag{11}$$

if $\gamma+2\ge n(2-m)/m^2$, or $p\ge 2/m$ otherwise. Then there exists $\epsilon>0$ such that for any initial data

$$(u_0,u_1)\in\mathcal{D}_m\,, \qquad \textit{satisfying} \quad \|(u_0,u_1)\|_{\mathcal{D}_m}\le\epsilon\,, \tag{12}$$

there exists a solution to (2). *Moreover, the solution and its energy satisfy the decay estimates*

$$\|u(t,\cdot)\|_{L^2} \lesssim (1+t)^{-n\left(\frac{1}{m}-\frac{1}{2}\right)}\,\|(u_0,u_1)\|_{\mathcal{D}_m}\,, \tag{13}$$

$$\|(\nabla u,u_t)(t,\cdot)\|_{L^2} \lesssim \begin{cases} (1+t)^{-n\left(\frac{1}{m}-\frac{1}{2}\right)-1}\,\|(u_0,u_1)\|_{\mathcal{D}_m} & \text{if } \mu>2+n(2/m-1),\\ (1+t)^{-\frac{\mu}{2}}\log(e+t)\,\|(u_0,u_1)\|_{\mathcal{D}_m} & \text{if } \mu=2+n(2/m-1). \end{cases} \tag{14}$$

Remark 1. We recall that in space dimension $n\ge3$ we assumed $p\le 1+2/(n-2)$ in Hypothesis 1.

For $n\ge3$, the set $(1+2(2+\gamma)/n, 1+2/(n-2)]$ of the global existence in Theorem 1 is nonempty if, and only if, either $\gamma\in[-2,-1]$, or $\gamma\in(-1,1)$ and $n<2(2+\gamma)/(1+\gamma)$.

For $n=3$, the range of admissible exponents p for the global existence in Theorem 2 is nonempty if, and only if, $\gamma<4$. We have the range $(1+(2+\gamma)/3,3]$ if $\gamma\in[1,4)$, and the range $[2,3]$ if $\gamma\in[-2,1)$. For $n=4$ we only have the admissible exponent $p=2$, provided that $\gamma<2$.

More in general, for any $m\in[1,2)$ there exists $\overline{n}=\overline{n}(m,\gamma)\ge3$ such that the range of admissible exponents is empty for $n\ge\overline{n}$. If $\gamma\in[-2,-1]$ then $\overline{n}(m,\gamma)\to\infty$ as $m\to2$.

Remark 2. Let us assume $\mu\ge n+2$ and let the data verify condition (7). We may compare Theorems 1, 2 and 3, looking for the largest range of admissible exponents p. Indeed, due to the bound $p\ge2$ in Theorem 2, we may get benefit by applying Theorem 3 for some $m\in(1,2)$, or even Theorem 1.
Let us fix $n\ge1$. If $\gamma\ge n-2$, then the range in Theorem 2 cannot be further improved, i.e we get

$$p\in\begin{cases} (1+(2+\gamma)/n,\infty) & \text{if } n=1,2 \text{ and } \gamma\ge n-2,\\ (1+(2+\gamma)/3,3] & \text{if } n=3 \text{ and } \gamma\in[3,4). \end{cases}$$

If $\gamma\in(-2,n-2)$, let $m\in(1,2)$ be the largest solution to

$$\left(\frac{2+\gamma}{n}\right)m^2+m-2=0\,.$$

In correspondence of this $m=m(n,\gamma)$, we obtain the range in Theorem 3, i.e. either $p>(1+(2+\gamma)m/n$ if $n=1,2$ or $p\in(1+(2+\gamma)m/n, 1+2/(n-2)]$, for any $n\ge3$ which makes the interval nonempty.

Finally, if $\gamma = -2$ we obtain either the range $p > 1$ if $n = 1, 2$, or the range $p \in (1, 1 + 2/(n-2))$ if $n \geq 3$, by applying Theorem 1.

If $\mu \in (2, n+2)$, we may apply Theorem 3 only for $m \in [\ell, 2)$, where

$$\ell = \ell(n, \mu) := \frac{2n}{n + \mu - 2}\,. \tag{15}$$

In particular, setting $m = \ell$ we immediately have the following.

Corollary 1. *Let $n \geq 1$ and $\mu \in (2, 2+n)$, and let us assume*

$$p > 1 + \frac{2(2+\gamma)}{n+\mu-2}\,, \tag{16}$$

$$\text{if} \quad \gamma \geq \frac{(\mu-2)(n+\mu-2)}{2n} - 2\,,$$

or $p \geq 1 + (\mu-2)/n$ otherwise. Let $\ell = \ell(n, \mu)$ be defined as in (15)*. Then there exists $\epsilon > 0$ such that for any initial data*

$$(u_0, u_1) \in \mathcal{D}_\ell\,, \qquad \text{satisfying} \quad \|(u_0, u_1)\|_{\mathcal{D}_\ell} \leq \epsilon\,, \tag{17}$$

there exists a solution to (2)*. Moreover, the solution and its energy satisfy the decay estimates*

$$\|u(t, \cdot)\|_{L^2} \lesssim (1+t)^{-(\frac{\mu}{2}-1)}\, \|(u_0, u_1)\|_{\mathcal{D}_\ell}\,, \tag{18}$$

$$\|(\nabla u, u_t)(t, \cdot)\|_{L^2} \lesssim (1+t)^{-\frac{\mu}{2}} \log(e+t)\, \|(u_0, u_1)\|_{\mathcal{D}_\ell}\,. \tag{19}$$

2. Models with time-dependent speed

More in general, one may investigate on the global existence for a wave equation with time-dependent propagation speed

$$\begin{cases} u_{tt} - \lambda(t)^2 \triangle u + b(t) u_t = f(t, u), \quad t \geq 0,\ x \in \mathbb{R}^n\,, \\ u(0, x) = u_0(x)\,, \\ u_t(0, x) = u_1(x)\,, \end{cases} \tag{20}$$

expecting interactions between the speed $\lambda(t)$ and the damping coefficient $b(t)$. In this setting, one may still classify the dissipation produced by the damping term in *effective* and *non effective*, with respect to the speed and to the considered estimate (see [1, 2]). In particular, we are interested in the following two models.

Example 1 (Polynomial speed). Let $\lambda(t) = (1+t)^{q-1}$ for some $q > 0$, and $b(t) = \nu(1+t)^{-1}$ for some $\nu \in \mathbb{R}$, that is,

$$\begin{cases} u_{tt} - (1+t)^{2(q-1)} \triangle u + \frac{\nu}{1+t}\, u_t = f(t, u), \quad t \geq 0,\ x \in \mathbb{R}^n\,, \\ u(0, x) = u_0(x)\,, \\ u_t(0, x) = u_1(x)\,. \end{cases} \tag{21}$$

With respect to this model, we will denote $\Lambda(t) = (1+t)^q/q$, and

$$\mu = \mu(\nu, q) := \frac{\nu - 1}{q} + 1\,.$$

We remark that for $q = 1$ we find again (2) and $\nu = \mu$.

Example 2 (Exponential speed). Let $\lambda(t)=e^{rt}$ for some $r>0$ and $b=\nu$ for some $\nu\in\mathbb{R}$, that is,

$$\begin{cases} u_{tt}-e^{2rt}\triangle u+\nu\,u_t=f(t,u), \quad t\geq 0,\ x\in\mathbb{R}^n\,,\\ u(0,x)=u_0(x)\,,\\ u_t(0,x)=u_1(x)\,. \end{cases} \tag{22}$$

With respect to this model, we will denote $\Lambda(t)=e^{rt}/r$, and

$$\mu=\mu(\nu):=\nu+1\,.$$

To deal with both models in Examples 1 and 2, we modify the assumption on $f(t,u)$.

Hypothesis 2. We assume that the nonlinear term in (20) satisfies

$$f(t,0)=0\,, \qquad |f(t,u)-f(t,v)|\lesssim \lambda(t)^2\Lambda(t)^\gamma\,|u-v|(|u|+|v|)^{p-1}\,, \tag{23}$$

for some $\gamma\geq -2$ and for a given $p>1$, satisfying $p\leq 1+2/(n-2)$ if $n\geq 3$.

With the notation in Examples 1 and 2, the inequality in condition (23) may be explicitated by means of the time-dependent speed $\lambda(t)$ and its anti-derivative $\Lambda(t)$, giving

$$|f(t,u)-f(t,v)|\lesssim (1+t)^{(\gamma+2)q-2}\,|u-v|(|u|+|v|)^{p-1}\,, \tag{24}$$

$$|f(t,u)-f(t,v)|\lesssim e^{(\gamma+2)rt}\,|u-v|(|u|+|v|)^{p-1}\,. \tag{25}$$

To state our results, we still use Notation 1 but now we refer to

$$\|(\lambda\nabla u,u_t)(t,\cdot)\|_{L^2}^2:=\lambda(t)^2\,\|\nabla u(t,\cdot)\|_{L^2}^2+\|u_t(t,\cdot)\|_{L^2}^2\,,$$

as the energy of the solution to (20).

Theorem 4. *Let $n\geq 1$, $\mu\geq 2$ and $p>1+2(2+\gamma)/n$. Then there exists $\epsilon>0$ such that, for any initial data as in* (4) *there exists a solution to* (20)*. Moreover, the solution and its energy satisfy the estimates*

$$\|u(t,\cdot)\|_{L^2}\lesssim \|(u_0,u_1)\|_{H^1\times L^2}\,, \tag{26}$$

$$\|(\lambda\nabla u,u_t)(t,\cdot)\|_{L^2}\lesssim \lambda(t)\,\Lambda(t)^{-1}\,\|(u_0,u_1)\|_{H^1\times L^2}\,. \tag{27}$$

Theorem 5. *Let $m\in[1,2)$ and $n\leq 4/(2-m)$. Let us assume* (10)*, and* (11) *if $\gamma+2\geq n(2-m)/m^2$, or $p\geq 2/m$ otherwise. Then there exists $\epsilon>0$ such that, for any initial data as in* (12) *there exists a solution to* (20)*. Moreover, the solution and its energy satisfy the estimates*

$$\|u(t,\cdot)\|_{L^2}\lesssim \Lambda(t)^{-n\left(\frac{1}{m}-\frac12\right)}\,\|(u_0,u_1)\|_{\mathcal{D}_m}\,, \tag{28}$$

$$\|(\lambda\nabla u,u_t)(t,\cdot)\|_{L^2}\lesssim \begin{cases} \lambda(t)\,\Lambda(t)^{-n\left(\frac{1}{m}-\frac12\right)-1}\,\|(u_0,u_1)\|_{\mathcal{D}_m} & \mu>2+n(2/m-1),\\ \lambda(t)\,\Lambda(t)^{-\frac{\mu}{2}}\log(e+\Lambda(t))\,\|(u_0,u_1)\|_{\mathcal{D}_m} & \text{if } \mu=2+n(2/m-1). \end{cases} \tag{29}$$

In the polynomial case the exponent $1+(2+\gamma)/n$ obtained in Theorem 5 for $m=1$ can be proved to be *critical* by using a *modified* test function method. Indeed, thanks to Theorem 1 in [3], there exists no global solution to (2) if $p\leq 1+(2+\gamma)/n$, for suitable, arbitrarily small data in L^1.

Remark 3. Taking $\lambda(t)=(1+t)^{q-1}$ as in Example 1 or, respectively, $\lambda(t)=e^{rt}$ as in Example 2, estimates (26)-(27) may be written in the form

$$\|u(t,\cdot)\|_{L^2}\lesssim \|(u_0,u_1)\|_{H^1\times L^2}\,,$$

$$\|\nabla u(t,\cdot)\|_{L^2}\lesssim (1+t)^{-q}\,\|(u_0,u_1)\|_{H^1\times L^2}\,,$$

$$\|u_t(t,\cdot)\|_{L^2} \lesssim (1+t)^{-1}\, \|(u_0,u_1)\|_{H^1\times L^2}\,,$$

or, respectively,

$$\begin{aligned}
\|u(t,\cdot)\|_{L^2} &\lesssim \|(u_0,u_1)\|_{H^1\times L^2}\,,\\
\|\nabla u(t,\cdot)\|_{L^2} &\lesssim e^{-rt}\, \|(u_0,u_1)\|_{H^1\times L^2}\,,\\
\|u_t(t,\cdot)\|_{L^2} &\lesssim \|(u_0,u_1)\|_{H^1\times L^2}\,.
\end{aligned}$$

Estimates (28)-(29) may be similarly written, including the additional decay rate $(1+t)^{-n\left(\frac{1}{m}-\frac{1}{2}\right)q}$ or, respectively, $e^{-n\left(\frac{1}{m}-\frac{1}{2}\right)rt}$.

Corollary 2. *Let $n\geq 1$ and μ,p be as in Corollary* 1. *Then there exists $\epsilon>0$ such that for any initial data as in* (17) *there exists a solution to* (20). *Moreover, the solution and its energy satisfy the decay estimates* (28)-(29) *with $m=\ell$, that is,*

$$\|u(t,\cdot)\|_{L^2} \lesssim \Lambda(t)^{-\left(\frac{\mu}{2}-1\right)}\, \|(u_0,u_1)\|_{\mathcal{D}_\ell}\,, \tag{30}$$

$$\|(\lambda\nabla u,u_t)(t,\cdot)\|_{L^2} \lesssim \lambda(t)\,\Lambda(t)^{-\frac{\mu}{2}}\log(e+\Lambda(t))\, \|(u_0,u_1)\|_{\mathcal{D}_\ell}\,. \tag{31}$$

Theorems 4 and 5 still hold if we consider a more general propagation speed, provided that we take a damping term in a suitable form.

Hypothesis 3. We assume that $\lambda\in\mathcal{C}^1$, with $\lambda(t)>0$ for any $t\geq 0$ and $\lambda\notin L^1$. Let

$$\Lambda(t) := \lambda_0 + \int_0^t \lambda(\tau)\,d\tau\,,$$

for some $\lambda_0>0$, be an anti-derivative of $\lambda(t)$. We assume that

$$b(t) := \mu\,\frac{\lambda(t)}{\Lambda(t)} - \frac{\lambda'(t)}{\lambda(t)}\,, \tag{32}$$

for some $\mu>0$, for any $t\geq 0$.

We remark that $\Lambda(t)$ is a strictly positive, strictly increasing function such that $\Lambda(t)\to\infty$ as $t\to\infty$. The assumption $\lambda\notin L^1$ which guarantees this latter property was first used in [5, 6] to derive energy estimates in the setting of linear systems, eventually with the presence of a dissipative lower order term. On the other hand, if we consider the equation

$$u_{tt} - \lambda(t)^2\triangle u + b(t)u_t = 0\,,$$

then still a dissipative effect on the energy $\|(\lambda\nabla u,u_t)\|_{L^2}$ appears (see [2]), provided that

$$\frac{\lambda'(t)}{\lambda(t)} + b(t) \geq 0\,. \tag{33}$$

We notice that (33) reduces to $\lambda'(t)\geq 0$ if $b\equiv 0$ (see [8]). Dealing with (20), thanks to the special structure of $b(t)$ given by (32) we see that (33) is satisfied for any $\mu\geq 0$.

Remark 4. It is clear that Hypothesis 3 is consistent with the notation used in Examples 1 and 2. On the other hand, polynomial and exponential speeds in Examples 1 and 2 have the following property: there exists an anti-derivative $\Lambda(t)$ of $\lambda(t)$ and a constant $\alpha\in\mathbb{R}$ such that

$$\frac{\lambda'(t)}{\lambda(t)} = \alpha\,\frac{\lambda(t)}{\Lambda(t)}\,. \tag{34}$$

Property (34) means that if $b(t) = \nu\lambda(t)/\Lambda(t)$ for some $\nu \in \mathbb{R}$, then (32) holds with $\mu = \nu + \alpha$. This constant is $\alpha = (q-1)/q$ in Example 1 and $\alpha = 1$ in Example 2. We notice that (34) is equivalent to say $\lambda(t) = C\,\Lambda(t)^\alpha$, for some $C > 0$.

Theorems 1-2-3 immediately follow as a consequence of Theorems 4-5, which we will prove in Section 4 for a general propagation speed and for the related dissipation, satisfying Hypothesis 3.

3. Linear Estimates

In order to prove our results we will apply Duhamel's principle. Therefore, we derive estimates for the family of parameter-dependent linear Cauchy problems:

$$\begin{cases} v_{tt} - \lambda(t)^2 \triangle v + b(t)\, v_t = 0, \quad t \geq s,\ x \in \mathbb{R}^n\,, \\ v(s,x) = v_0(x)\,, \\ v_t(s,x) = v_1(x)\,. \end{cases} \tag{35}$$

Lemma 3. *Let $(v_0, v_1) \in L^2 \times L^2$. If $\mu \geq 1$ then the solution to (35) satisfies the estimate*

$$\|v(t,\cdot)\|_{L^2} \lesssim \|v_0\|_{L^2} + \frac{\Lambda(s)}{\lambda(s)}\, \|v_1\|_{L^2}\,. \tag{36}$$

Moreover, if $(v_0, v_1) \in H^1 \times L^2$ and $\mu \geq 2$, then the energy of the solution to (35) satisfies the estimate

$$\|(\lambda\nabla v, v_t)(t,\cdot)\|_{L^2} \lesssim \frac{\lambda(t)}{\Lambda(t)}\, \Lambda(s)\, \left(\|v_0\|_{H^1} + \frac{1}{\lambda(s)}\, \|v_1\|_{L^2} \right)\,. \tag{37}$$

Lemma 4. *Let $(v_0, v_1) \in L^m \cap L^2$ for some $m \in [1,2)$. If $\mu \geq 1$ and $\mu > n(2/m-1)$ then the solution to (35) satisfies the estimate*

$$\|v(t,\cdot)\|_{L^2} \lesssim \Lambda(t)^{-n\left(\frac{1}{m}-\frac{1}{2}\right)} \left\{ \|v_0\|_{L^m} + \frac{\Lambda(s)}{\lambda(s)}\, \|v_1\|_{L^m} + \Lambda(s)^{n\left(\frac{1}{m}-\frac{1}{2}\right)} \left(\|v_0\|_{L^2} + \frac{\Lambda(s)}{\lambda(s)}\, \|v_1\|_{L^2} \right) \right\}\,, \tag{38}$$

whereas if $\mu = n(2/m-1) \geq 1$ it satisfies the estimate

$$\|v(t,\cdot)\|_{L^2} \lesssim \Lambda(t)^{-\frac{\mu}{2}}\, \log\left(1 + \frac{\Lambda(t)}{\Lambda(s)}\right) \left\{ \|v_0\|_{L^m} + \frac{\Lambda(s)}{\lambda(s)}\, \|v_1\|_{L^m} + \Lambda(s)^{\frac{\mu}{2}} \left(\|v_0\|_{L^2} + \frac{\Lambda(s)}{\lambda(s)}\, \|v_1\|_{L^2} \right) \right\}\,. \tag{39}$$

Moreover, if $(v_0, v_1) \in \mathcal{D}_m$ and $\mu > 2 + n(2/m-1)$ then the energy of the solution to (35) satisfies the estimate

$$\begin{aligned} \|(\lambda\nabla v, v_t)(t,\cdot)\|_{L^2} \lesssim \lambda(t)\, \Lambda(t)^{-n\left(\frac{1}{m}-\frac{1}{2}\right)-1} & \left\{ \|v_0\|_{L^m} + \frac{\Lambda(s)}{\lambda(s)}\, \|v_1\|_{L^m} \right. \\ & \left. + \Lambda(s)^{n\left(\frac{1}{m}-\frac{1}{2}\right)+1} \left(\|v_0\|_{H^1} + \frac{1}{\lambda(s)}\, \|v_1\|_{L^2} \right) \right\}\,, \end{aligned} \tag{40}$$

whereas if $\mu = 2 + n(2/m-1)$ it satisfies the estimate

$$\begin{aligned} \|(\lambda\nabla v, v_t)(t,\cdot)\|_{L^2} \lesssim \lambda(t)\, \Lambda(t)^{-\frac{\mu}{2}}\, \log\left(1 + \frac{\Lambda(t)}{\Lambda(s)}\right) & \left\{ \|v_0\|_{L^m} + \frac{\Lambda(s)}{\lambda(s)}\, \|v_1\|_{L^m} \right. \\ & \left. + \Lambda(s)^{\frac{\mu}{2}} \left(\|v_0\|_{H^1} + \frac{1}{\lambda(s)}\, \|v_1\|_{L^2} \right) \right\}\,. \end{aligned} \tag{41}$$

We recall that taking $\lambda(t) = 1$, $\Lambda(t) = 1 + t$ and $b(t) = \mu(1+t)^{-1}$ we obtain the linear estimates corresponding to (2).

Remark 5. Since (35) is linear, we may write the solution to (35) into the form

$$v(t,x) = E_0(t,s,x) *_{(x)} v_0(x) + E_1(t,s,x) *_{(x)} v_1(x)\,. \tag{42}$$

The estimates in Lemmas 3 and 4 are deeply related to the special structure of the equation in (35). To prove them we follow the approach used in [25] to derive $L^2 - L^2$ estimates for the linear damped wave equation

$$u_{tt} - \triangle u + \frac{\mu}{1+t}\, u_t = 0\,, \qquad t \geq 0\,,$$

modifying it to derive $(L^m \cap L^2) - L^2$ estimates, and taking into account the presence of the parameter s and of the speed $\lambda(t)$.

Let us put $w(\Lambda(t)\,|\xi|) = \widehat{v}(t,\xi)$, and let us denote $\tau = \Lambda(t)\,|\xi|$ and $\sigma = \Lambda(s)\,|\xi|$. Then $\sigma > 0$ for any $\xi \neq 0$, and from the equation in (35) we obtain the ordinary differential equation

$$w'' + w + \frac{\mu}{\tau}\, w' = 0, \qquad \tau \geq \sigma\,. \tag{43}$$

If we put $\rho := (1-\mu)/2$ and $w(\tau) = \tau^\rho\, y(\tau)$ then from (43) we obtain the Bessel's differential equation of order $\pm\rho$:

$$\tau^2 y'' + \tau y' + (\tau^2 - \rho^2) y = 0\,, \qquad \tau \geq \sigma\,. \tag{44}$$

A system of linearly independent solution to (44) is given by the pair of Hankel functions $\mathcal{H}^\pm_\rho(\tau)$, hence we put

$$w^\pm(\tau) := \tau^\rho \mathcal{H}^\pm_\rho(\tau)\,.$$

If we define

$$\Psi_{k,r,\delta}(t,s,|\xi|) := \frac{i\pi}{4}\,|\xi|^k\, \det \begin{pmatrix} \mathcal{H}^-_r\big(\Lambda(s)\,|\xi|\big) & \mathcal{H}^-_{r+\delta}\big(\Lambda(t)\,|\xi|\big) \\ \mathcal{H}^+_r\big(\Lambda(s)\,|\xi|\big) & \mathcal{H}^+_{r+\delta}\big(\Lambda(t)\,|\xi|\big) \end{pmatrix} \tag{45}$$

$$\equiv -\frac{\pi}{2}\, \csc(\rho\pi)\, |\xi|^k\, \det \begin{pmatrix} \mathcal{I}^-_{-r}\big(\Lambda(s)\,|\xi|\big) & \mathcal{I}^-_{-(r+\delta)}\big(\Lambda(t)\,|\xi|\big) \\ (-1)^{|\delta|}\mathcal{I}^+_r\big(\Lambda(s)\,|\xi|\big) & \mathcal{I}^+_{r+\delta}\big(\Lambda(t)\,|\xi|\big) \end{pmatrix}\,, \tag{46}$$

then the solution to (35) is given by

$$\widehat{v}(t,\xi) = \Phi_0(t,s,\xi)\widehat{v_0}(\xi) + \Phi_1(t,s,\xi)\widehat{v_1}(\xi)\,,$$

that is, $\Phi_j(t,s,\xi)$ is the Fourier transform of $E_j(t,s,x)$ introduced in (42). We may now write the multipliers and their time-derivatives in the form

$$\Phi_0(t,s,\xi) = \frac{\Lambda(t)^\rho}{\Lambda(s)^{\rho-1}}\, \Psi_{1,\rho-1,1}\,, \tag{47}$$

$$\Phi_1(t,s,\xi) = -\frac{1}{\lambda(s)}\, \frac{\Lambda(t)^\rho}{\Lambda(s)^{\rho-1}}\, \Psi_{0,\rho,0}\,, \tag{48}$$

$$\partial_t \Phi_0(t,s,\xi) = \lambda(t)\, \frac{\Lambda(t)^\rho}{\Lambda(s)^{\rho-1}}\, \Psi_{2,\rho-1,0}\,, \tag{49}$$

$$\partial_t \Phi_1(t,s,\xi) = -\frac{\lambda(t)}{\lambda(s)}\, \frac{\Lambda(t)^\rho}{\Lambda(s)^{\rho-1}}\, \Psi_{1,\rho,-1}\,, \tag{50}$$

Let us fix $K \in (0,1)$, independent on s and t. The following three properties hold:

$$|\mathcal{H}^\pm_\nu(\tau)| \lesssim \tau^{-1/2}\,, \quad \text{for } \tau \in [K,\infty), \tag{51}$$

$$|\mathcal{H}^\pm_\nu(\tau)| \lesssim \begin{cases} \tau^{-|\nu|}\,, & \text{for } \tau \in (0,K] \text{ if } \nu \neq 0, \\ -\log\tau\,, & \text{for } \tau \in (0,K] \text{ if } \nu = 0, \end{cases} \tag{52}$$

$$|\mathcal{I}^\pm_\nu(\tau)| \lesssim \tau^\nu\,, \quad \text{for } \tau \in (0,\infty). \tag{53}$$

According to the parameter $s \geq 0$ and to the variable $t \geq s$, we divide the frequencies in three intervals:

$$I_1 := \left\{|\xi| \geq \frac{K}{\Lambda(s)}\right\}, \qquad I_2 := \left\{\frac{K}{\Lambda(s)} \geq |\xi| \geq \frac{K}{\Lambda(t)}\right\}, \qquad I_3 := \left\{\frac{K}{\Lambda(t)} \geq |\xi|\right\}.$$

We are now ready to prove our linear estimates.

Proof of Lemma 3. By virtue of Parseval's identity, to derive $L^2 - L^2$ estimates for the solution to (35) and its energy, it is sufficient to control the L^∞ norm of $|\xi|^k \partial_t^l \Phi_j(t,s,\xi)$ for $l+k = 0,1$ and $j = 0,1$, which expressions may be obtained by (47)-(48)-(49)-(50).

In the interval I_1 it holds $\tau \geq \sigma \geq K$, therefore thanks to (51) we get

$$|\Psi_{k,r,\delta}(t,s,|\xi|)| \lesssim |\xi|^k\,(\Lambda(s)|\xi|)^{-1/2}\,(\Lambda(t)|\xi|)^{-1/2}.$$

It immediately follows that

$$\Psi_{1,\rho-1,1}\,, \quad |\xi|\Psi_{0,\rho,0}\,, \quad |\xi|^{-1}\,\Psi_{2,\rho-1,0}\,, \quad \Psi_{1,\rho,-1}\,,$$

are all bounded by $\Lambda(s)^{-1/2}\,\Lambda(t)^{-1/2}$. On the other hand, we can estimate

$$|\Psi_{0,\rho,0}| \lesssim |\xi|^{-1}\,\Lambda(s)^{-1/2}\,\Lambda(t)^{-1/2} \lesssim \Lambda(s)^{1/2}\,\Lambda(t)^{-1/2}.$$

In the interval I_2 it holds $\tau \geq K \geq \sigma$, therefore thanks to (51) and (52) we get

$$|\Psi_{k,r,\delta}(t,s,|\xi|)| \lesssim |\xi|^k\,(\Lambda(s)|\xi|)^{-|r|}\,(\Lambda(t)|\xi|)^{-1/2},$$

hence it follows

$$\begin{aligned}
|\Psi_{1,\rho-1,1}| &\lesssim |\xi|\,(\Lambda(s)|\xi|)^{-|\rho-1|}\,(\Lambda(t)|\xi|)^{-1/2},\\
|\Psi_{0,\rho,0}| &\lesssim (\Lambda(s)|\xi|)^{-|\rho|}\,(\Lambda(t)|\xi|)^{-1/2},\\
|\xi|\,|\Psi_{1,\rho-1,1}|,\ |\Psi_{2,\rho-1,0}| &\lesssim |\xi|^2\,(\Lambda(s)|\xi|)^{-|\rho-1|}\,(\Lambda(t)|\xi|)^{-1/2},\\
|\xi|\,|\Psi_{0,\rho,0}|,\ |\Psi_{1,\rho,-1}| &\lesssim |\xi|\,(\Lambda(s)|\xi|)^{-|\rho|}\,(\Lambda(t)|\xi|)^{-1/2}.
\end{aligned}$$

Using $|\xi|^{-1} \lesssim \Lambda(t)$ and $\mu \geq 1$, that is, $\rho \leq 0$, one can estimate

$$\begin{aligned}
|\Psi_{1,\rho-1,1}| &\lesssim |\xi|^{-(1/2-\rho)}\,\Lambda(s)^{\rho-1}\,\Lambda(t)^{-1/2} \lesssim \Lambda(s)^{\rho-1}\,\Lambda(t)^{-\rho},\\
|\Psi_{0,\rho,0}| &\lesssim |\xi|^{-(1/2-\rho)}\Lambda(s)^{\rho}\,\Lambda(t)^{-1/2} \lesssim \Lambda(s)^{\rho}\,\Lambda(t)^{-\rho}.
\end{aligned}$$

If $\mu \geq 2$, that is, $\rho \leq -1/2$, then

$$\begin{aligned}
|\xi|\,|\Psi_{1,\rho-1,1}|, |\Psi_{2,\rho-1,0}| &\lesssim |\xi|^{\rho+1/2}\,\Lambda(s)^{\rho-1}\,\Lambda(t)^{-1/2} \lesssim \Lambda(s)^{\rho-1}\,\Lambda(t)^{-\rho-1},\\
|\xi|\,|\Psi_{0,\rho,0}|, |\Psi_{1,\rho,-1}| &\lesssim |\xi|^{\rho+1/2}\,\Lambda(s)^{\rho}\,\Lambda(t)^{-1/2} \lesssim \Lambda(s)^{\rho}\,\Lambda(t)^{-\rho-1}.
\end{aligned}$$

In the interval I_3 it holds $K \geq \tau \geq \sigma$. We use (46) and (53), obtaining

$$\begin{aligned}
|\Psi_{k,r,\delta}(t,s,|\xi|)| &\lesssim |\xi|^k\,\Big((\Lambda(s)|\xi|)^{-r}\,(\Lambda(t)|\xi|)^{r+\delta} + (\Lambda(s)|\xi|)^{r}\,(\Lambda(t)|\xi|)^{-(r+\delta)}\Big)\\
&= |\xi|^{k+\delta}\,\Lambda(s)^{-r}\,\Lambda(t)^{r+\delta} + |\xi|^{k-\delta}\,\Lambda(s)^{r}\,\Lambda(t)^{-(r+\delta)}\\
&\lesssim \Lambda(s)^{-r}\,\Lambda(t)^{r-k} + \Lambda(s)^{r}\,\Lambda(t)^{-r-k} \lesssim \Lambda(s)^{-|r|}\,\Lambda(t)^{|r|-k},
\end{aligned}$$

provided that $k \geq |\delta|$, since $|\xi| \lesssim \Lambda(t)^{-1}$ and $\Lambda(s) \leq \Lambda(t)$. Since $\rho \leq 0$, using $|\xi| \lesssim \Lambda(t)^{-1}$ where needed, it follows again

$$\begin{aligned}
|\Psi_{1,\rho-1,1}| &\lesssim |\xi|\Lambda(s)^{\rho-1}\,\Lambda(t)^{1-\rho} \lesssim \Lambda(s)^{\rho-1}\,\Lambda(t)^{-\rho},\\
|\Psi_{0,\rho,0}| &\lesssim \Lambda(s)^{\rho}\,\Lambda(t)^{-\rho},
\end{aligned}$$

$$|\xi|\,|\Psi_{1,\rho-1,1}|, |\Psi_{2,\rho-1,0}| \lesssim |\xi|^2\,\Lambda(s)^{\rho-1}\,\Lambda(t)^{1-\rho} \lesssim \Lambda(s)^{\rho-1}\,\Lambda(t)^{-\rho-1}\,,$$
$$|\xi|\,|\Psi_{0,\rho,0}|, |\Psi_{1,\rho,-1}| \lesssim |\xi|\,\Lambda(s)^{\rho}\,\Lambda(t)^{-\rho} \lesssim \Lambda(s)^{\rho}\,\Lambda(t)^{-\rho-1}\,.$$

Using $\Lambda(s)\leq\Lambda(t)$ and $\rho\leq 1/2$, in I_1 we also have

$$\Lambda(s)^{-1/2}\,\Lambda(t)^{-1/2} \leq \Lambda(s)^{\rho-1}\,\Lambda(t)^{-\rho}\,,$$
$$\Lambda(s)^{1/2}\,\Lambda(t)^{-1/2} \leq \Lambda(s)^{\rho}\,\Lambda(t)^{-\rho}\,.$$

Summarizing and recalling (47)-(48), estimate (36) follows. If $\rho\leq -1/2$, that is, $\mu\geq 2$, then

$$\Lambda(s)^{-1/2}\,\Lambda(t)^{-1/2} \leq \Lambda(s)^{\rho}\,\Lambda(t)^{-\rho-1}\,.$$

Recalling (47)-(48)-(49)-(50), the proof of (37) follows. □

Proof of Lemma 4. We follow the proof of Lemma 3 with some modifications. In I_1 we notice that

$$\frac{\Lambda(t)^{\rho}}{\Lambda(s)^{\rho-1}}\,\Lambda(s)^{1/2}\,\Lambda(t)^{-1/2} = \Lambda(s)^{\frac{\mu}{2}+1}\,\Lambda(t)^{-\frac{\mu}{2}}\,,$$
$$\frac{\Lambda(t)^{\rho}}{\Lambda(s)^{\rho-1}}\,\Lambda(s)^{-1/2}\,\Lambda(t)^{-1/2} = \Lambda(s)^{\frac{\mu}{2}}\,\Lambda(t)^{-\frac{\mu}{2}}\,.$$

Moreover, since $\Lambda(s)\leq\Lambda(t)$ we may estimate

$$\Lambda(s)^{\frac{\mu}{2}+1}\,\Lambda(t)^{-\frac{\mu}{2}} \leq \Lambda(t)^{-n\left(\frac{1}{m}-\frac{1}{2}\right)}\Lambda(s)^{1+n\left(\frac{1}{m}-\frac{1}{2}\right)} \quad \text{if } \mu\geq n(2/m-1),$$
$$\Lambda(s)^{\frac{\mu}{2}}\,\Lambda(t)^{-\frac{\mu}{2}} \leq \begin{cases} \Lambda(t)^{-n\left(\frac{1}{m}-\frac{1}{2}\right)}\Lambda(s)^{n\left(\frac{1}{m}-\frac{1}{2}\right)} & \text{if } \mu\geq n(2/m-1),\\ \Lambda(t)^{-n\left(\frac{1}{m}-\frac{1}{2}\right)-1}\Lambda(s)^{n\left(\frac{1}{m}-\frac{1}{2}\right)+1} & \text{if } \mu\geq 2+n(2/m-1). \end{cases}$$

Let us define $q:=(1/m-1/2)^{-1}\in[2,\infty)$. By virtue of Parseval's identity, we may now estimate

$$\|v(t,s,\cdot)\|_{L^2} \lesssim \sum_{j=0}^{1}\left(\|\Phi_j(t,s,\xi)\|_{L^\infty(I_1)}\,\|v_j(t,s,\cdot)\|_{L^2} + \|\Phi_j(t,s,\xi)\|_{L^q(I_2\cup I_3)}\,\|v_j(t,s,\cdot)\|_{L^m}\right)\,,$$

and similarly for the energy. Let

$$J_2^{\pm} := \int_{|\xi|\in I_2} |\xi|^{q\,(\rho\pm 1/2)}d\xi\,, \qquad J_3^{\pm} := \int_{|\xi|\in I_3} |\xi|^{q\,(j+k\pm\delta)}d\xi\,,$$

and $\eta:=\Lambda(t)|\xi|$. It follows

$$J_2^{\pm} \lesssim \Lambda(t)^{-q\,(\rho\pm 1/2)-n}\int_{|\eta|\geq K}|\eta|^{q\,(\rho\pm 1/2)}d\eta \lesssim \Lambda(t)^{-q\,(\rho\pm 1/2)-n}\,,$$
$$J_3^{\pm} \lesssim \Lambda(t)^{-q\,(j+k\pm\delta)-n}\int_{|\eta|\leq K}|\eta|^{q\,(j+k\pm\delta)}d\eta \lesssim \Lambda(t)^{-q\,(j+k\pm\delta)-n}\,,$$

provided that $q\,(\rho\pm 1/2)<-n$ and that $j+k\pm\delta>-n$. Therefore we obtain

$$\|\Psi_{1,\rho-1,1}\|_{L^q(I_2\cup I_3)} \lesssim \Lambda(s)^{\rho-1}\,\Lambda(t)^{-\rho-n/q}\,,$$
$$\|\Psi_{0,\rho,0}\|_{L^q(I_2\cup I_3)} \lesssim \Lambda(s)^{\rho}\,\Lambda(t)^{-\rho-n/q}\,,$$

provided that $\rho-1/2<-n/q$, that is, $\mu>2n(1/m-1/2)$, and

$$\|(\xi\,\Psi_{1,\rho-1,1},\Psi_{2,\rho-1,0})\|_{L^q(I_2\cup I_3)} \lesssim \Lambda(s)^{\rho-1}\,\Lambda(t)^{-\rho-1-n/q}\,,$$
$$\|(\xi\,\Psi_{0,\rho,0},\Psi_{1,\rho,-1})\|_{L^q(I_2\cup I_3)} \lesssim \Lambda(s)^{\rho}\,\Lambda(t)^{-\rho-1-n/q}\,,$$

provided that $\rho+1/2<-n/q$, i.e. $\mu>2+2n(1/m-1/2)$. If $\mu=1+n(2/m-1)\pm 1$, the estimate of $J_2^{\pm}$ gives

$$|J_2^{\pm}| \leq C_n\left(\log(K/\Lambda(s))-\log(K/\Lambda(t))\right)\,,$$

Combining the estimates for high and low frequencies, we conclude the proof. □

4. Proof of Theorems 4 and 5

We will use the linear estimates (38) and (40) to prove (28) and (29) for $\mu > 2 + n(2/m - 1)$. The special case $\mu = 2 + n(2/m - 1)$ can be easily proved by replacing estimate (40) with (41), whereas estimates (26) and (27) follow from (36) and (37).

Using Duhamel's principle and (42), a function $u \in \mathcal{C}([0,\infty), H^1) \cap \mathcal{C}^1([0,\infty), L^2)$ is a solution to (20) if, and only if, it is a fixed point for the operator N given by

$$Nu(t,x) = E_0(t,0,x) *_{(x)} u_0(x) + E_1(t,0,x) *_{(x)} u_1(x) + \int_0^t E_1(t,s,x) *_{(x)} f(s,u(s,x))\, ds\,, \tag{54}$$

i.e. $Nu(t,\cdot) = u(t,\cdot)$ in H^1 and $\partial_t Nu(t,\cdot) = u_t(t,\cdot)$ in L^2, for any $t \in [0,\infty)$. For any $t \geq 0$, we consider the spaces

$$X(t) := \mathcal{C}([0,t], H^1) \cap \mathcal{C}^1([0,t], L^2)\,, \qquad X_0(t) = \mathcal{C}([0,t], H^1)\,,$$

with the norms

$$\|w\|_{X(t)} := \sup_{0\leq\tau\leq t} \Lambda(\tau)^{n(1/m-1/2)} \Big(\|w(\tau,\cdot)\|_{L^2} + \Lambda(\tau)\|\nabla w(\tau,\cdot)\|_{L^2} + \lambda(\tau)^{-1}\Lambda(\tau)\|w_t(\tau,\cdot)\|_{L^2} \Big)\,,$$
$$\|w\|_{X_0(t)} := \sup_{0\leq\tau\leq t} \Lambda(\tau)^{n(1/m-1/2)} \Big(\|w(\tau,\cdot)\|_{L^2} + \Lambda(\tau)\|\nabla w(\tau,\cdot)\|_{L^2} \Big)\,.$$

We claim that for any data $(u_0,u_1) \in \mathcal{D}_m$ the operator N satisfies the estimates

$$\|Nu\|_{X(t)} \leq C\, \|(u_0,u_1)\|_{\mathcal{D}_m} + C\|u\|_{X_0(t)}^p\,, \tag{55}$$
$$\|Nu - N\tilde{u}\|_{X(t)} \leq C\|u-\tilde{u}\|_{X_0(t)} \big(\|u\|_{X_0(t)}^{p-1} + \|\tilde{u}\|_{X_0(t)}^{p-1} \big)\,, \tag{56}$$

for any $u, \tilde{u} \in X(t)$, uniformly with respect to $t \in [0,\infty)$.
If (55) and (56) hold, then N maps $X(t)$ into itself and there exists a unique fixed point $u \in X(t)$ for the operator N, for sufficiently small data. Indeed, let $\epsilon := \|(u_0,u_1)\|_{\mathcal{D}_m}$, and let us define the sequence $u^{(j)} = Nu^{(j-1)}$ for any $j \geq 1$, with $u^{(0)} = 0$. Thanks to (55), there exists $\epsilon_0 = \epsilon_0(C) > 0$, such that

$$\|u^{(j)}\|_{X(t)} \leq 2C\epsilon, \tag{57}$$

for any $\epsilon \in [0,\epsilon_0]$. Moreover, let us fix $\epsilon_0(C)$ be such that $C\epsilon_0^{p-1} < 1$. Using (56) and (57), we obtain

$$\|u^{(j+1)} - u^{(j)}\|_{X(t)} \leq C\epsilon^{p-1}\|u^{(j)} - u^{(j-1)}\|_{X(t)}\,, \tag{58}$$

therefore $\{u^{(j)}\}$ is a Cauchy sequence in the Banach space $X(t)$, converging to the unique solution of $Nu = u$. Since the constants are independent of t, the global existence follows. The definition of $\|u\|_{X(t)}$ leads to the decay estimates (28)-(29).

Therefore, we only need to prove our claims (55) and (56). During the proof a special role will be played by different applications of Gagliardo-Nirenberg inequality:

$$\|u(s,\cdot)\|_{L^q}^p \lesssim \|u(s,\cdot)\|_{L^2}^{p(1-\theta(q))} \|\nabla u(s,\cdot)\|_{L^2}^{p\theta(q)}, \qquad \text{where} \tag{59}$$
$$\theta(q) := n\left(\frac{1}{2} - \frac{1}{q}\right)\,, \qquad \text{for any} \quad q \in \left[2\,,\, \frac{2n}{n-2}\right]\,. \tag{60}$$

We prove (55), being the proof of (58) completely analogous.

Proof of (55). From (38)-(40) we derive

$$\begin{aligned}\|Nu(t,\cdot)\|_{L^2} &\lesssim \Lambda(t)^{-n(\frac{1}{m}-\frac{1}{2})}\,\|(u_0,u_1)\|_{L^m\times L^2}\\ &\quad+\Lambda(t)^{-n(\frac{1}{m}-\frac{1}{2})}\int_0^t \lambda(s)^{-1}\,\Lambda(s)\,\|f(s,u(s,\cdot))\|_{L^m}ds\\ &\quad+\Lambda(t)^{-n(\frac{1}{m}-\frac{1}{2})}\int_0^t \lambda(s)^{-1}\,\Lambda(s)^{1+n(\frac{1}{m}-\frac{1}{2})}\,\|f(s,u(s,\cdot))\|_{L^2}ds\,, \qquad (61)\\ \|(\lambda\nabla Nu,\partial_t Nu)(t,\cdot)\|_{L^2} &\lesssim \lambda(t)\,\Lambda(t)^{-n(\frac{1}{m}-\frac{1}{2})-1}\,\|(u_0,u_1)\|_{\mathcal{D}_m}\\ &\quad+\lambda(t)\,\Lambda(t)^{-n(\frac{1}{m}-\frac{1}{2})-1}\int_0^t \lambda(s)^{-1}\,\Lambda(s)\,\|f(s,u(s,\cdot))\|_{L^m}ds\\ &\quad+\lambda(t)\,\Lambda(t)^{-n(\frac{1}{m}-\frac{1}{2})-1}\int_0^t \lambda(s)^{-1}\,\Lambda(s)^{1+n(\frac{1}{m}-\frac{1}{2})}\,\|f(s,u(s,\cdot))\|_{L^2}ds\,. \qquad (62)\end{aligned}$$

By using (23) we can estimate $|f(s,u)|\lesssim\lambda(s)^2\Lambda(s)^\gamma\,|u|^p$. Since $p\geq 2/m$, and $p\leq n/(n-2)$ if $n\geq 3$, we can apply (59) with $q=mp$ and $q=2p$, obtaining

$$\||u(s,\cdot)|^p\|_{L^m}\lesssim\|u(s,\cdot)\|_{L^{mp}}^p\lesssim\|u\|_{X_0(s)}^p\Lambda(s)^{-p(n(1/m-1/2)+\theta(mp))}=\|u\|_{X_0(s)}^p\Lambda(s)^{-\frac{n}{m}(p-1)}\,, \qquad (63)$$

$$\||u(s,\cdot)|^p\|_{L^2}\lesssim\|u(s,\cdot)\|_{L^{2p}}^p\lesssim\|u\|_{X_0(s)}^p\Lambda(s)^{-p(n(1/m-1/2)+\theta(2p))}=\|u\|_{X_0(s)}^p\Lambda(s)^{-\frac{pn}{m}+\frac{n}{2}}\,. \qquad (64)$$

We notice that:

$$1+n\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{pn}{m}+\frac{n}{2}+\gamma=1-\frac{n}{m}(p-1)+\gamma\,,$$

hence

$$\begin{aligned}\|Nu(t,\cdot)\|_{L^2} &\lesssim \Lambda(t)^{-n(\frac{1}{m}-\frac{1}{2})}\,\|(u_0,u_1)\|_{L^m\cap L^2}\\ &\quad+\|u\|_{X_0(t)}^p\,\Lambda(t)^{-n(\frac{1}{m}-\frac{1}{2})}\int_0^t\lambda(s)\,\Lambda(s)^{1-\frac{n}{m}(p-1)+\gamma}ds \qquad (65)\\ \|(\lambda\nabla Nu,\partial_t Nu)(t,\cdot)\|_{L^2} &\lesssim \lambda(t)\,\Lambda(t)^{-n(\frac{1}{m}-\frac{1}{2})-1}\,\|(u_0,u_1)\|_{\mathcal{D}_m}\\ &\quad+\|u\|_{X_0(t)}^p\,\lambda(t)\,\Lambda(t)^{-n(\frac{1}{m}-\frac{1}{2})-1}\int_0^t\lambda(s)\,\Lambda(s)^{1-\frac{n}{m}(p-1)+\gamma}\,ds\,. \qquad (66)\end{aligned}$$

Thanks to (11), if we put $r=\Lambda(s)$ then we get

$$\int_0^t\lambda(s)\,\Lambda(s)^{1-\frac{n}{m}(p-1)+\gamma}\,ds=\int_{\Lambda(0)}^{\Lambda(t)}r^{1-\frac{n}{m}(p-1)+\gamma}\,dr\leq C\,,$$

and this concludes the proof of (55). □

5. Data from a weighted energy space

If $f=f(u)$, we may overcome the lower bound $p\geq 2$ in Theorem 2 if we assume smallness of the initial data in some weighted energy space. Similarly in Theorem 5 with $m=1$.
Let $\lambda(t)$ and $b(t)$ satisfy Hypothesis 3. For any $t\geq 0$, we define the exponential weight

$$\omega_{(t)}(x):=\exp\left(\frac{\mu}{2}\,\frac{|x|^2}{\Lambda(t)^2}\right)\,, \qquad (67)$$

and we denote by $L^2(\omega_{(t)})$ and $H^1(\omega_{(t)})$ the weighted spaces with norms:

$$\|u\|_{L^2(\omega_{(t)})}^2:=\int_{\mathbb{R}^n}|u(x)|^2\,\omega_{(t)}^2(x)\,dx\,,\qquad \|u\|_{H^1(\omega_{(t)})}^2=\|u\|_{L^2(\omega_{(t)})}^2+\|\nabla u\|_{L^2(\omega_{(t)})}^2\,.$$

One may easily check that $L^2(\omega_{(t)})\hookrightarrow L^1\cap L^2$, for any $\mu>0$ and $t\geq 0$.

Theorem 6. *Let $n \geq 1$, $\mu \geq n+2$. Let $f(t,u) = \lambda(t)^2 f_1(u)$, with $f_1(u)$ satisfying*

$$f_1(0) = 0\,, \qquad |f_1(u) - f_1(v)| \lesssim |u-v|(|u|+|v|)^{p-1}\,,$$

for some $p > 1+2/n$, and $p \leq 1+2/(n-2)$ if $n \geq 3$. Then there exists $\epsilon > 0$ such that for any initial data

$$(u_0,u_1) \in H^1(\omega_{(0)}) \times L^2(\omega_{(0)})\,, \qquad \textit{satisfying} \quad \|(u_0,u_1)\|_{H^1(\omega_{(0)})\times L^2(\omega_{(0)})} \leq \epsilon\,, \tag{68}$$

there exists a solution u to (20)*. Moreover, $u \in \mathcal{C}([0,\infty), H^1(\omega_{(t)})) \cap \mathcal{C}^1([0,\infty), L^2(\omega_{(t)}))$, and*

$$\begin{aligned}
\|u(t,\cdot)\|_{L^2} &\lesssim \Lambda(t)^{-\frac{n}{2}} \|(u_0,u_1)\|_{L^2(\omega_{(0)})}\,,\\
\|(\lambda\nabla u, u_t)(t,\cdot)\|_{L^2} &\lesssim \lambda(t)\Lambda(t)^{-\frac{n}{2}-1} \|(u_0,u_1)\|_{H^1(\omega_{(0)})\times L^2(\omega_{(0)})}\,,\\
\|u(t,\cdot)\|_{L^2(\omega_{(t)})} &\lesssim \lambda(t)\,\Lambda(t)\,\|(u_0,u_1)\|_{H^1(\omega_{(0)})\times L^2(\omega_{(0)})}\,,\\
\|(\lambda\nabla u, u_t)(t,\cdot)\|_{L^2(\omega_{(t)})} &\lesssim \lambda(t)\,\|(u_0,u_1)\|_{H^1(\omega_{(0)})\times L^2(\omega_{(0)})}\,.
\end{aligned}$$

The range of admissible exponents p for the global existence in Theorem 6 is nonempty for any $n \geq 1$. If we consider (2), then we assume $f = f(u)$, and the weight is given by

$$\omega_{(t)}(x) := \exp\left(\frac{\mu}{2}\,\frac{|x|^2}{1+t^2}\right)\,.$$

By assuming compactly supported data, Y. Wakasugi recently extended the result in [16] to prove that if $f(u) = |u|^p$ with $p > 1+2/n$ then there exists $\overline{\mu} = \overline{\mu}(p,n)$ satisfying $\overline{\mu}(p,n) \approx n^2\,(p-(1+2/n))^{-2}$ such that for any $\mu \geq \overline{\mu}$ there exists a global solution to (2). A loss of information in the decay estimates like $(1+t)^\epsilon$ also appears, where $\epsilon \approx \mu^{-1}$ (see [24]). We remark that in Theorem 6 we do not require compact support, the threshold is $\mu \geq n+2$ for any $p > 1+2/n$, and we do not have loss of information in the decay estimates with respect to the linear problem. Moreover, we can deal with a more general propagation speed $\lambda(t)$.

In order to prove Theorem 6, we follow the approach in [4, 11]. For the sake of brevity, we only sketch the main ideas, highlighting the differences due to the presence of the propagation speed $\lambda(t)$.

One can easily prove the local existence of the solution to (20) in

$$\mathcal{C}\big([0,T_{\max}), H^1(\omega_{(t)})\big) \cap \mathcal{C}\big([0,T_{\max}), L^2(\omega_{(t)})\big)\,,$$

for any $p \leq 1+2/(n-2)$, where by $T_{\max} > 0$ we denote the maximal existence time. Moreover,

$$\limsup_{t\to T_{\max}} \left(\|u(t,\cdot)\|^2_{H_1(\omega_{(t)})} + \lambda(t)^{-2}\,\|u_t(t,\cdot)\|^2_{L^2(\omega_{(t)})}\right) = \infty\,, \tag{69}$$

if $T_{\max} < \infty$. Let us define the function

$$\psi(t,x) := \log\omega_{(t)}(x) = \frac{\mu}{2}\,\frac{|x|^2}{\Lambda(t)^2}\,,$$

which has the following property:

$$\mu\,\frac{\lambda(t)}{\Lambda(t)}\,\psi_t(t,x) = -|\lambda(t)\nabla\psi(t,x)|^2\,, \qquad \text{in particular } \psi_t(t,x) \leq 0 \text{ since } \mu \geq 0. \tag{70}$$

We are now in a position to prove the following.

Lemma 5. *Let u be the local solution to* (20)*. Then for any $t \in [0,T_{\max})$ and for any $\varepsilon \in (0, 2-2/(p+1))$, the following energy estimate holds:*

$$\|(\lambda\nabla u, u_t)(t,\cdot)\|^2_{L^2(\omega_{(t)})} \leq C\,\lambda(t)^2\,\left(\|(u_0,u_1)\|^2_{H^1(\omega_{(0)})\times L^2(\omega_{(0)})} + \|(u_0,u_1)\|^{\frac{p+1}{2}}_{H^1(\omega_{(0)})\times L^2(\omega_{(0)})}\right)$$

$$+ C_\varepsilon\, \lambda(t)^2 \sup_{s\in[0,t]} \Big(\Lambda(s)^\varepsilon \| e^{(\varepsilon+2/(p+1))\psi(s,\cdot)} u(s,\cdot)\|_{L^{p+1}}\Big)^{p+1} .$$

Proof. We recall that $f(t,u)=\lambda(t)^2 f_1(u)$ in Theorem 6. If we define the functional

$$G(t) := \frac{1}{\lambda(t)^2}\, \|(\lambda\nabla u, u_t)(t,\cdot)\|^2_{L^2(\omega_{(t)})} - \int_{\mathbb{R}^n} F(u)\,dx\,, \quad \text{where } F(u) := \int_0^u f_1(v)\,dv\,,$$

then it follows that

$$G(t)-G(0) \leq -4 \int_0^t \int_{\mathbb{R}^n} \psi_t(s,x)\, e^{2\psi(s,x)}\, F(u(s,x))\,dx\,ds\,. \tag{71}$$

Indeed, we have:

$$\partial_t \left(\frac{e^{2\psi}}{2} \big(\lambda(t)^{-2}\, |u_t|^2 + |\nabla u|^2 - F(u) \big)\right) = \nabla\cdot (e^{2\psi} u_t \nabla u) + \lambda(t)^{-2} \psi_t e^{2\psi} u_t^2$$
$$+ \frac{e^{2\psi}}{\psi_t} |u_t\nabla\psi - \psi_t \nabla u|^2 - \lambda(t)^{-2} \frac{e^{2\psi}}{\psi_t} u_t^2 \big((b(t)+\lambda'(t)/\lambda(t))\psi_t + |\nabla\psi|^2\big) - 2\psi_t e^{2\psi} F(u)\,.$$

By using divergence theorem and (70), the proof of (71) follows. By using Sobolev embedding, we get

$$G(0) \lesssim \|(u_0,u_1)\|^2_{H^1(\omega_{(0)})\times L^2(\omega_{(0)})} + \|(u_0,u_1)\|^{\frac{p+1}{2}}_{H^1(\omega_{(0)})\times L^2(\omega_{(0)})}\,.$$

Estimating

$$|\psi_t(s,x)| e^{-\varepsilon(p+1)\psi(s,x)} = 2\, \frac{\lambda(t)}{\Lambda(t)} \psi(s,x) e^{-\varepsilon(p+1)\psi(s,x)} \leq C_\varepsilon\, \frac{\lambda(t)}{\Lambda(t)}\,, \qquad \text{and} \qquad \int_0^t \frac{\lambda(s)}{\Lambda(s)^{1+\varepsilon}}\,ds \leq C_\varepsilon\,,$$

and $|F(u(s,x))| \lesssim |u(s,x)|^{p+1}$ we may conclude the proof. $\square$

The advantage of working with weighted spaces relies in the chance to estimate

$$\|f_1(u(s,\cdot))\|_{L^1} \lesssim \|u(s,\cdot)\|^p_{L^p} \lesssim \Lambda(s)^{\frac{n}{2}}\, \|e^{\varepsilon\psi(s,\cdot)} u(s,\cdot)\|^p_{L^{2p}}\,, \tag{72}$$

by using Hölder inequality and

$$\int_{\mathbb{R}^n} e^{-\frac{c|x|^2}{\Lambda(s)^2}}\,dx = \Lambda(s)^n \int_{\mathbb{R}^n} e^{-c|y|^2} dy \lesssim \Lambda(s)^n\,.$$

Trivially, we may also estimate

$$\|f_1(u(s,\cdot))\|_{L^2} \lesssim \|e^{\varepsilon\psi(s,\cdot)} u(s,\cdot)\|^p_{L^{2p}}\,. \tag{73}$$

Proof of Theorem 6. By contradiction, let us assume that for any $\epsilon>0$ there exist data satisfying (68) such that the solution to (20) is not global, that is, $T_{\max}<\infty$. Similarly to the proof of Theorem 5, for any $t\in(0,T_{\max})$ we may consider the space

$$X(t) := \mathcal{C}\big([0,t], H^1(\omega_{(\tau)})\big) \cap \mathcal{C}\big([0,t], L^2(\omega_{(\tau)})\big)\,, \qquad \text{with norm}$$
$$\|u\|_{X(t)} := \max_{\tau\in[0,t]} \Big(\lambda(\tau)^{-1} \|(\lambda\nabla u, u_t)(\tau,\cdot)\|_{L^2(\omega_{(\tau)})} \tag{74}$$
$$+ \lambda(\tau)^{-1} \Lambda(\tau)^{\frac{n}{2}+1} \|(\lambda\nabla u, u_t)(\tau,\cdot)\|_{L^2} + \Lambda(\tau)^{\frac{n}{2}} \|u(\tau,\cdot)\|_{L^2}\Big)\,. \tag{75}$$

We may immediately use Lemma 5 to estimate the weighted energy in (74). On the other hand, using the linear estimates in Lemma 4 as we did in the proof of Theorem 5, together with (72)-(73), we can control the terms in (75), obtaining:

$$\|u\|_{X(t)} \lesssim \epsilon + \epsilon^{\frac{p+1}{2}} + \sup_{\tau\in[0,t]} \Big(\Lambda(\tau)^\varepsilon\, \|e^{(\varepsilon+2/(p+1))\psi(\tau,\cdot)} u(\tau,\cdot)\|_{L^{p+1}}\Big)^{\frac{p+1}{2}}$$

$$+ \sup_{\tau\in[0,t]} \left(\Lambda(\tau)^{\frac{n}{2}+\varepsilon} \|e^{\varepsilon\psi(\tau,\cdot)}u(\tau,\cdot)\|_{L^{2p}}\right)^p . \tag{76}$$

In order to manage the last two terms we use a Gagliardo-Nirenberg type inequality (see Lemma 2.3 in [11] and Lemma 9 in [4]) and we get

$$\|e^{\sigma\psi(t,\cdot)}v\|_{L^q} \leq C_\sigma\, \Lambda(t)^{1-\theta(q)}\, \|\nabla v\|_{L^2}^{1-\sigma}\, \|e^{\psi(t,\cdot)}\nabla v\|_{L^2}^{\sigma} , \tag{77}$$

for any $\sigma \in [0,1]$ and $v \in H^1_{\sigma\psi(t,\cdot)}$, where $\theta(q)$ is as in (60). By using (77), it follows

$$\|e^{(\varepsilon+2/(p+1))\psi(\tau,\cdot)}u(\tau,\cdot)\|_{L^{p+1}} \leq \|u\|_{X(t)}\, \Lambda(\tau)^{1-\theta(p+1)-(1-2/(p+1)-\varepsilon)(n/2+1)} , \tag{78}$$

$$\|e^{\varepsilon\psi(\tau,\cdot)}u(\tau,\cdot)\|_{L^{2p}} \leq \|u\|_{X(t)}\, \Lambda(\tau)^{1-\theta(2p)-(1-\varepsilon)(n/2+1)} . \tag{79}$$

We remark that $2 < p+1 < 2p \leq 2n/(n-2)$, hence Gagliardo-Nirenberg inequality is applicable. Since $p > 1 + 2(2+\gamma)/n$, it follows that

$$1-\theta(p+1)-(1-2/(p+1))(n/2+1) = 1-\theta(2p)-(n/2+1) = \frac{1-(p-1)n/2}{p} < 0 .$$

Therefore, if we take $\varepsilon > 0$ sufficiently small, from (76) we may obtain

$$\|u\|_{X(t)} \lesssim \epsilon + \epsilon^{\frac{p+1}{2}} + \|u\|_{X(t)}^{\frac{p+1}{2}} + \|u\|_{X(t)}^p ,$$

uniformly with respect to $t \in [0, T_{\max})$. By standard arguments, it follows that $\|u\|_{X(t)}$ is bounded with respect to $t \in [0, T_{\max})$, provided that $\epsilon > 0$ is sufficiently small. Hence $\|u(t,\cdot)\|_{L^2(\omega_{(t)})}$ is bounded too. This contradicts (69), hence the maximal existence time is $T_{\max} = \infty$. □

Appendix A. Linear estimates under the threshold $\mu = 2$

If $\mu \in (0,2)$ then the $L^2 - L^2$ estimate of the energy of the solution to the linear problem (35) is worse than (37), since the dissipation becomes *non effective* and we get

$$\|(\lambda\nabla v, v_t)(t,\cdot)\|_{L^2} \lesssim \lambda(t)\, \Lambda(t)^{-\frac{\mu}{2}}\, \Lambda(s)^{\frac{\mu}{2}} \left(\|v_0\|_{H^1} + \frac{1}{\lambda(s)}\|v_1\|_{L^2}\right) . \tag{80}$$

Indeed, we may follow the proof of Lemma 4, but now $\rho \in (-1/2, 1/2)$. The estimate in I_1 remains the same. In I_2, using $|\xi| \lesssim \Lambda(s)^{-1}$, we get

$$|\xi|\,|\Psi_{1,\rho-1,1}|,\ |\Psi_{2,\rho-1,0}| \lesssim |\xi|^{\rho+1/2}\,\Lambda(s)^{\rho-1}\,\Lambda(t)^{-1/2} \lesssim \Lambda(s)^{-3/2}\,\Lambda(t)^{-1/2} ,$$

$$|\xi|\,|\Psi_{0,\rho,0}|,\ |\Psi_{1,\rho,-1}| \lesssim \begin{cases} |\xi|^{\rho+1/2}\,\Lambda(s)^{\rho}\,\Lambda(t)^{-1/2} \lesssim \Lambda(s)^{-1/2}\,\Lambda(t)^{-1/2} , & \text{if } \mu \in (1,2), \\ |\xi|^{1/2-\rho}\,\Lambda(s)^{-\rho}\,\Lambda(t)^{-1/2} \lesssim \Lambda(s)^{-1/2}\,\Lambda(t)^{-1/2} , & \text{if } \mu \in (0,1), \end{cases}$$

If $\rho \in (0,1/2)$, i.e. $\mu \in (0,1)$, using $|\xi| \lesssim \Lambda(t)^{-1}$, we derive

$$|\xi|\,|\Psi_{0,\rho,0}| \lesssim |\xi|\,\Lambda(s)^{-\rho}\,\Lambda(t)^{\rho} \lesssim \Lambda(s)^{-\rho}\,\Lambda(t)^{\rho-1} ,$$

$$|\Psi_{1,\rho,-1}| \lesssim \Lambda(s)^{-\rho}\,\Lambda(t)^{\rho-1} ,$$

in the interval I_3. Since $|\rho| - 1 \leq -1/2$, the worst rate for $|\xi|\,|\Psi_{1,\rho-1,1}|$, $|\Psi_{2,\rho-1,0}|$, $|\xi|\,|\Psi_{0,\rho,0}|$ and $|\Psi_{1,\rho,-1}|$ is now given by $\Lambda(t)^{-1/2}$, therefore, due to

$$\frac{\Lambda(t)^{\rho}}{\Lambda(s)^{\rho-1}}\,\Lambda(s)^{-1/2}\,\Lambda(t)^{-1/2} = \Lambda(s)^{\frac{\mu}{2}}\,\Lambda(t)^{-\frac{\mu}{2}} ,$$

estimates (80) follows. Estimate (80) is consistent with the energy estimate proved in Example 3 in [2] for $s = 0$ and $\mu \in [0,2]$.

One may immediately use estimate (80) to extend Theorem 4 to the case $\mu \in [1,2)$, modifying the proof where needed.

Remark 6. Let $n\geq 1$. If $\mu\in[1,2)$ and

$$p>1+\frac{4(2+\gamma)}{\mu\, n}\,, \tag{81}$$

then there exists $\epsilon>0$ such that for any initial data satisfying (4) there exists a solution to (20). Moreover, the solution satisfies (26) and its energy satisfies the estimate

$$\|(\lambda\nabla u,u_t)(t,\cdot)\|_{L^2}\lesssim \lambda(t)\,\Lambda(t)^{-\frac{\mu}{2}}\,\|(u_0,u_1)\|_{H^1\times L^2}\,. \tag{82}$$

However, we do not expect condition (81) to be optimal. Indeed, for $\mu\in(0,2)$ the model becomes more *hyperbolic* hence the use of linear L^2-L^2 estimates which are analogous to the corresponding heat equation is not meaningful (see [25]).

A different effect appears if we are interested in estimates of the solution to (35), for $\mu\in(0,1)$. It is convenient to separate contributions coming from v_0 and v_1. Let $v_1\equiv 0$. If $v_0\in H^1$ or $v_0\in L^m\cap H^1$, we still have estimates (36) for any $\mu\geq 0$, estimate (38) for $\mu>n(2/m-1)$ and estimate (39) for $\mu=n(2/m-1)$. Otherwise, the estimate rate with respect to t becomes worse.

Lemma 6. *Let $\mu\in(0,1)$ and $v_0\equiv 0$. If $v_1\in L^2$ then the solution to* (35) *satisfies the estimate*

$$\|v(t,\cdot)\|_{L^2}\lesssim \Lambda(t)^{1-\mu}\,\frac{\Lambda(s)^\mu}{\lambda(s)}\,\|v_1\|_{L^2}\,. \tag{83}$$

If $v_1\in L^m\cap L^2$ for some $m\in[1,2)$ and $\mu<2-n(2/m-1)$, then the solution to (35) *satisfies the estimate*

$$\|v(t,\cdot)\|_{L^2}\lesssim \Lambda(t)^{(1-\mu)-n\left(\frac{1}{m}-\frac12\right)}\,\frac{\Lambda(s)^\mu}{\lambda(s)}\,\Big(\|v_1\|_{L^m}+\Lambda(s)^{n\left(\frac1m-\frac12\right)}\,\|v_1\|_{L^2}\Big)\,, \tag{84}$$

whereas if $\mu=2-n(2/m-1)$, it satisfies the estimate

$$\|v(t,\cdot)\|_{L^2}\lesssim \Lambda(t)^{-\frac{\mu}{2}}\,\frac{1}{\lambda(s)}\,\log\left(1+\frac{\Lambda(t)}{\Lambda(s)}\right)\,\Big(\Lambda(s)^\mu\,\|v_1\|_{L^m}+\Lambda(s)^{1+\frac{\mu}{2}}\,\|v_1\|_{L^2}\Big)\,, \tag{85}$$

Proof. We only prove (84), being the other two estimates similar. We follow the proof of Lemma 4, but now $\rho\in(0,1/2)$. The estimate in I_1 remains the same. In I_2 we may estimate

$$|\Psi_{0,\rho,0}|\lesssim|\xi|^{-\rho-1/2}\Lambda(s)^{-\rho}\Lambda(t)^{-1/2}\,,$$

therefore, using $q(-\rho-1/2)<-n$, that is, $\mu<2-n(2/m-1)$, we derive

$$\int_{|\xi|\in I_2}|\xi|^{-q(\rho+1/2)}\,d\xi\lesssim\Lambda(t)^{q(\rho+1/2)-n}\,.$$

On the other hand, in I_3 we may estimate $|\Psi_{0,\rho,0}|\lesssim\Lambda(s)^{-\rho}\Lambda(t)^{\rho}$, therefore

$$\int_{|\xi|\in I_3}1\,d\xi\lesssim\Lambda(t)^{-n}\,.$$

Summarizing, we proved

$$\|\Psi_{0,\rho,0}\|_{L^q(I_2\cap I_3)}\lesssim\Lambda(s)^{-\rho}\,\Lambda(t)^{\rho-n/q}\,,$$

hence estimate (84) follows. ∎

Appendix B. Additional considerations in one space dimension

In this Appendix we fix $n=1$.

If $\mu\in[2,3)$, according to Corollary 2, if data are small in $\mathcal{D}_\ell$ then we have global existence for any $p\geq\mu-1$ satisfying (16), i.e.

$$p>1+\frac{2(2+\gamma)}{\mu-1}\,.$$

If $\mu\in[1,2)$, according to Remark 6, if data are small in $H^1\times L^2$ then we have global existence for any

$$p>1+\frac{4(2+\gamma)}{\mu}\,.$$

However, we may improve this lower bound for p if data are small in $\mathcal{D}_1$.

Corollary 7. *Let $n=1$, $\mu\in[1,3)$ and $p\geq2$, satisfying*

$$p>1+\frac{4(2+\gamma)}{\mu+1}\,. \tag{86}$$

Then for any initial data satisfying (7) *there exists a solution to* (20). *Moreover, estimate* (28) *with $m=1$ holds for the solution, together with*

$$\|(\lambda\nabla u,u_t)(t,\cdot)\|_{L^2}\lesssim\begin{cases}\lambda(t)\,\Lambda(t)^{-\frac{\mu}{2}}\log(e+\Lambda(t))\,\|(u_0,u_1)\|_{\mathcal{D}_1} & \text{if } \mu\in(2,3),\\ \lambda(t)\,\Lambda(t)^{-\frac{\mu}{2}}\,\|(u_0,u_1)\|_{\mathcal{D}_1}\,, & \text{if } \mu\in[1,2],\end{cases} \tag{87}$$

for its energy.

We remark that the exponent in (86) is lower than the one in (16) for any $\mu\in[2,3)$, and it is lower than the one in (81) for any $\mu\in[1,2)$. This improvement does not appear in space dimension $n\geq2$, if one extends this strategy.

Proof. We prove for $\mu\in(2,3)$, being the case $\mu\in[1,2]$ analogous and simpler. We follow the proof of Theorem 5 but we consider the norm on $X_0(t)$ given by

$$\|w\|_{X_0(t)}:=\sup_{0\leq\tau\leq t}\left(\Lambda(\tau)^{\frac12}\|w(\tau,\cdot)\|_{L^2}+\Lambda(\tau)^{\frac{\mu}{2}}\big(\log(e+\Lambda(\tau))\big)^{-1}\|\nabla w(\tau,\cdot)\|_{L^2}\right),$$

and similarly the norm on $X(t)$. Using (59), we may estimate

$$\begin{aligned}\|u(s,\cdot)\|_{L^q}&\lesssim\|u\|_{X_0(s)}^p\Lambda(s)^{-(1-\theta(q))\frac{n}{2}-\theta(q)\frac{\mu}{2}}\big(\log(e+\Lambda(\tau))\big)^{\theta(q)}\\&=\Lambda(s)^{-\frac12-\theta(q)\frac{\mu-1}{2}}\big(\log(e+\Lambda(\tau))\big)^{\theta(q)}\,,\end{aligned} \tag{88}$$

for $q=p,\ell p,2p$, that is, (63)-(64) are replaced by

$$\begin{aligned}\|f(u(s,\cdot))\|_{L^1}&\lesssim\|u\|_{X_0(s)}^p\Lambda(s)^{\gamma-p(\frac1m-\frac12+\theta(p)\frac{\mu-1}{2})}=\|u\|_{X_0(s)}^p\Lambda(s)^{\gamma-p\frac14(\mu+1)+\frac12(\mu-1)}\,,\\ \|f(u(s,\cdot))\|_{L^\ell}&\lesssim\|u\|_{X_0(s)}^p\Lambda(s)^{\gamma-p(\frac1m-\frac12+\theta(\ell p)\frac{\mu-1}{2})}=\|u\|_{X_0(s)}^p\Lambda(s)^{\gamma-p\frac14(\mu+1)+\frac{1}{2\ell}(\mu-1)}\,,\\ \|f(u(s,\cdot))\|_{L^2}&\lesssim\|u\|_{X_0(s)}^p\Lambda(s)^{\gamma-p(\frac1m-\frac12+\theta(2p)\frac{\mu-1}{2})}=\|u\|_{X_0(s)}^p\Lambda(s)^{\gamma-p\frac{n}{4}(\mu+1)+\frac14(\mu-1)}\,.\end{aligned}$$

Let us put

$$p_r:=p\,\frac14\,(\mu+1)-\frac{1}{2r}\,(\mu-1)\,,\qquad r=1,\ell,2\,.$$

Using (38) with $m=1$ and (41) with $m=\ell$ we obtain

$$\begin{aligned}\|Nu(t,\cdot)\|_{L^2}&\lesssim\Lambda(t)^{-\frac12}\,\|(u_0,u_1)\|_{L^m\cap L^2}\\&\quad+\|u\|_{X_0(t)}^p\,\Lambda(t)^{-\frac12}\int_0^t\lambda(s)\,\Lambda(s)^{1+\gamma-p_1}\big(\log(e+\Lambda(\tau))\big)^p ds\end{aligned} \tag{89}$$

$$+ \|u\|_{X_0(t)}^p \Lambda(t)^{-\frac{1}{2}} \int_0^t \lambda(s)\,\Lambda(s)^{1+\frac{1}{2}+\gamma-p_2} \big(\log(e+\Lambda(\tau))\big)^p\,ds\,, \tag{90}$$

$$\|\nabla Nu(t,\cdot)\|_{L^2} \lesssim \Lambda(t)^{-\frac{\mu}{2}} \log(e+\Lambda(t))\, \|(u_0,u_1)\|_{\mathcal{D}_m}$$

$$+ \|u\|_{X_0(t)}^p \Lambda(t)^{-\frac{\mu}{2}} \log(e+\Lambda(t)) \int_0^t \lambda(s)\,\Lambda(s)^{1+\gamma-p_\ell} \big(\log(e+\Lambda(\tau))\big)^p\,ds \tag{91}$$

$$+ \|u\|_{X_0(t)}^p \Lambda(t)^{-\frac{\mu}{2}} \int_0^t \lambda(s)\,\Lambda(s)^{\frac{\mu}{2}+\gamma-p_2} \big(\log(e+\Lambda(\tau))\big)^p\,ds\,, \tag{92}$$

and similarly for $\partial_t Nu$. We notice that

$$p_\ell > p_1 > p_2 - \frac{1}{2}\,, \qquad \text{and that} \quad p_2 + 1 - \frac{\mu}{2} > p_2 - \frac{1}{2} = (p-1)\frac{1}{4}(\mu+1)\,,$$

therefore the integrals in (89)-(90)-(91)-(92) are bounded if, and only if, $(p_2 - 1/2) > 2+\gamma$, that is, (86). $\square$

We remark that in space dimension $n = 1$ the classical semilinear wave equation $u_{tt} - \triangle u = |u|^p$ admits no global solution, for any $p > 1$. Therefore, we still have concrete benefits from the damping term, even below the threshold $\mu = 2$. Moreover, if $\mu \in (0,1]$, one may use the linear estimate (85) to obtain global existence by assuming smallness of the initial data in $\mathcal{D}_\kappa$, where

$$\kappa(\mu) := \frac{2}{3-\mu}\,,$$

for any $p \geq 4/(3-\mu)$ such that

$$p > 1 + \frac{2(2+\gamma)}{\mu}\,. \tag{93}$$

In [24] it is proved that if $\mu \in (0,1)$ and $f = f(u) = |u|^p$, then there exists no global solution to (2) for any

$$1 < p \leq 1 + \frac{2}{n-(1-\mu)}\,, \tag{94}$$

provided that $u_1 \in L^1$ and

$$\int_{\mathbb{R}^n} u_1(x)\,dx > 0\,.$$

We notice that the exponent in (94) tends to Fujita exponent $1+2/n$ as $\mu \to 1$ and to Kato exponent $1+2/(n-1)$ (see [14, 21]) as $\mu \to 0$. This effect is related to the loss of parabolic properties of the equation in (2) as μ becomes smaller, in particular under the threshold $\mu = 1$. Following the proof of Theorem 1.4 in [24], condition (94) can be easily extended to

$$1 < p \leq 1 + \frac{2+\gamma}{n-(1-\mu)}\,.$$

if $f(t,u) \gtrsim (1+t)^\gamma |u|^p$. This exponent gives $1 + (2+\gamma)/\mu$ in space dimension $n = 1$. Still, there exists a gap between the exponents in (93) and (94). The problem to cover this gap remains open.

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